Logic

3. The Logic of *if, or & not-both* I

Overview

- Two Basic Ways of Thinking about this Logic
  - Stoic Modes
  - The Propositional Calculus
- Four Basic Forms of Argument
  - Modus Ponens
  - Modus Tollens
  - Disjunctive Syllogism
  - Conjunctive Syllogism
- Miscellaneous Comments

Some Arguments

- "There was a tradition in later sources that Prodicus [a Sophist] died in Athens by drinking the hemlock apparently after condemnation for 'corrupting the young'. This is … probably rightly dismissed as involving a confusion between Prodicus & Socrates—if it had been true, we would surely hear much more about it in earlier sources."
  —G. B. Kerferd, *The Sophistic Movement*, p. 46
- "[Since he's Irish], either his father is Irish or his mother is. But his mother is not Irish. So, his father is."

Two Formalizations of this Logic

- Megarian-Stoic Syllogisms (C3-C2 BC)
- The Inference Rules of the Propositional Calculus
  - based on Alfred North Whitehead & Bertrand Russell, *Principia Mathematica* (1910-13)

Stoic Syllogisms

- Syllogism
  - "A discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so"—Aristotle
- Stoic Syllogisms
  - begin with [i.e., have as premises] two propositions
    1. a compound proposition stating a logical relation between two propositions (if, or, not-both)
       - Compound propositions are propositions all of whose constituents are themselves propositions
         - » e.g., "If the budget is not balanced, then the nation is doomed."
         - » but not "John said toads are cuter than frogs."
    - the assertion or denial of one of the component propositions
  - end with [i.e., have as a conclusion]
    - the assertion or denial of the other of the component propositions

Three Basic Logical Relations

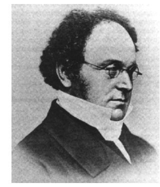
Name	Form	Symbolization	Names of Parts
Hypothetical (Conditional)	if … then …	$(P \rightarrow Q)$ or $(P \supset Q)$	Antecedent & Consequent
Disjunction	… or …	$(P \vee Q)$	Disjuncts
Denial of a Conjunction	not both … and …	$(P \uparrow Q)$ or $\sim(P \wedge Q)$	Conjuncts

Four Basic Modes

Classical Mode Name	*Modus ponendo ponens*	*Modus tollendo tollens*	*Modus tollendo ponens*	*Modus ponendo tollens*
Modern Mode Name	Modus ponens	Modus tollens	Disjunctive Syllogism	Conjunctive Syllogism
Premise	If *p* then *q*	If *p* then *q*	Either *p* or *q*	Not both *p* and *q*
Premise	*p*	not *q*	not *p*	*p*
Conclusion	*q*	not *p*	*q*	not *q*

The Propositional Calculus: What is a Calculus?

- The definition a calculus (cf. L. Susan Stebbing, *A Modern Introduction to Logic*, p. 180)
  - An instrument of reasoning whose purpose is to economize thought by providing a mechanical method of obtaining results, which can then be interpreted in a manner analogous to the way in which a mathematical equation can be interpreted
- The use of a calculus (*ibid.*)
  - "by means of such economy important discoveries may be made that would otherwise have been beyond the reach of finite minds."
  - cf. DeMorgan's Challenge:
    - Anyone who doubts the value of a calculus may be invited to answer the simple question: What people are not the descendants of those who are not my ancestors?
  - A calculus becomes important as propositions become more complex.



Wffs

- The prerequisite of a calculus
  - A distinction between well-formed formulae (wffs) & ill-formed ones
- The standard definition of a wff for the Propositional Calculus (PC)
  - any letter is a wff
  - a wff preceded by a ~ is a wff
  - two wffs joined by one of these symbols $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$ (or $\cdot$ $\vee$ $\supset$ $\equiv$) and surrounded by parentheses is a wff
- Interpreted for logic
  - any *propositional variable* is a wff
  - a wff preceded by a *not* is a wff
  - two wffs joined by one of these connectives *and, or, if, iff (=if & only if)* and surrounded by parentheses is a wff

The Utility of Recursivity

- This makes possible the analysis of complex propositions:

If either P or Q, then R.	$((P \vee Q) \rightarrow R)$
If P, then either Q or R-and-S.	$((P \rightarrow (Q \vee (R \wedge S)))$
Either P-and-Q or if R then S.	$((P \wedge Q) \vee (R \rightarrow S))$

Logical Connectives

- The four connectives
  - If $(P \supset Q)$ $(P \rightarrow Q)$
  - And $(P \cdot Q)$ $(P \wedge Q)$
  - Or $(P \vee Q)$
  - If & only if (logical equivalence) $(P \equiv Q)$ $(P \leftrightarrow Q)$
- This is a list driven by practicality
  - More are possible
    - E.g., $(P \leftarrow Q)$, equivalent to $(Q \rightarrow P)$
  - Less are sufficient
    - E.g., $(P \wedge Q)$ could be written $\sim(\sim P \vee \sim Q)$
- Note that (intuitively) …
  - in one, $(P \rightarrow Q)$, order matters
    - $(P \rightarrow Q)$ is different from $(Q \rightarrow P)$
  - in the others it does not
    - $(P \vee Q)$ is equivalent to $(Q \vee P)$

Hypothetical Syllogisms: Recognizing Modus Ponens

The key to M. ponens is *matching the right part of the conditional*, not the absence of the negation sign

These are all Modus ponens

$(P \rightarrow Q)$	$(P \rightarrow \sim Q)$	$(\sim P \rightarrow Q)$	$(\sim P \rightarrow \sim Q)$
P	P	$\sim P$	$\sim P$
$\therefore Q$	$\therefore \sim Q$	$\therefore Q$	$\therefore \sim Q$

These are *not* Modus ponens

$(\sim P \rightarrow Q)$	$(P \rightarrow \sim Q)$	$(P \rightarrow Q)$
P	P	Q
$\therefore$ —	$\therefore Q$	$\therefore P$

Hypothetical Syllogisms: Recognizing Modus Tollens

The key to M. ponens is *mismatching the right part of the conditional*, not the presence of the negation sign.

These are all Modus tollens

$(P \rightarrow Q)$	$(P \rightarrow {\sim}Q)$	$({\sim}P \rightarrow Q)$	$({\sim}P \rightarrow {\sim}Q)$
${\sim}Q$	Q	${\sim}Q$	Q
$\therefore {\sim}P$	$\therefore {\sim}P$	$\therefore P$	$\therefore P$

These are *not* Modus tollens

$(P \rightarrow {\sim}Q)$	$({\sim}P \rightarrow Q)$
${\sim}Q$	${\sim}P$
$\therefore$ –	$\therefore Q$

Hypothetical Syllogisms: Recognizing Disjunctive Syllogisms

The key to Disjunctive Syllogism is *mismatching (either part)*, not the presence of the negation sign. Also, drawing the correct conclusion.

These are all Disjunctive Syllogisms

$(P \vee Q)$	$(P \vee Q)$	$({\sim}P \vee Q)$	$({\sim}P \vee Q)$
${\sim}P$	${\sim}Q$	P	${\sim}Q$
$\therefore Q$	$\therefore P$	$\therefore Q$	$\therefore {\sim}P$

These are *not* Disjunctive Syllogisms

$(P \vee Q)$	$({\sim}P \vee Q)$
P	${\sim}P$
$\therefore {\sim}Q$	$\therefore Q$

Hypothetical Syllogisms: Recognizing Conjunctive Syllogisms

The key to Conjunctive Syllogism is *matching (either part)*, not the absence of the negation sign. Also, drawing the correct conclusion.

These are all Disjunctive Syllogisms

$(P \uparrow Q)$	$(P \uparrow Q)$	$({\sim}P \uparrow Q)$	$({\sim}P \uparrow {\sim}Q)$
P	Q	${\sim}P$	${\sim}P$
$\therefore {\sim}Q$	$\therefore {\sim}P$	$\therefore {\sim}Q$	$\therefore Q$

These are *not* Disjunctive Syllogisms

$({\sim}P \uparrow Q)$	$(P \uparrow Q)$
P	${\sim}P$
$\therefore Q$	$\therefore Q$

Distinguishing Conditional from Arguments

- An *argument* connects a premise & a conclusion:
  - "You don't study hard.
    So, you'll do badly on examinations."
  - Analysis
    - Premise: You don't study hard.
    - Conclusion: You'll do badly on examinations.
  - Here both the premise & the conclusion are asserted.
- A conditional (or hypothetical) statement connects an antecedent & a consequent:
  - "If you don't study hard, you'll do badly on you examination"
  - Analysis
    - Antecedent: You don't study hard.
    - Consequent: You'll do badly on examinations.
  - Here neither the antecedent nor the consequent is asserted.
    - It only asserts a logical connection between the two.
    - Or, what else would be true if the antecedent were.
  - Conditionals are useful because we often want to state a consequence of a proposition before we know whether the proposition is true.
    - E.g., in scientific work we may not know whether a theory is true
    - The best way to find out is to draw some predictions (consequences) from it and see whether the predictions correspond with experimental or observational results.

Disjunctive Syllogisms: Two Meanings of *or*

- Is this valid?
  1. Either John studied German in college or he studied Russian.
  2. He did study German.
  3. So, he did not study Russian.
- That depends how we understand the word *or*. It has two senses:
  - An inclusive sense: "at least one", "one or both"
    - E.g., (Speaking about someone who got into a very selective college) "Either he had very high test scores or he had very good letters of recommendation from his teachers."
    - Such a person, the speaker says, must have had at least one, but there's no reason to think that the student could not have had both.
  - An exclusive sense: "at least one", "one or both"
    - E.g., "Either Senator Obama will be the Democratic nominee or Senator Clinton will."
    - They cannot both be the nominee.
  - Some arguments that are invalid on the inclusive sense are valid on the exclusive sense.

Symbolization of "Or"

- The Symbols
  - Inclusive *or* $(P \vee Q)$
  - Exclusive *or* $(P \veebar Q)$ or $((P \vee Q) \wedge (P \uparrow Q))$
- The Choice
  - First, note that both senses of "or" include the connection $(P \vee Q)$.
    - So, *any time* one has two statements connected by an "or" one should write at least that.
  - Second, one should only add $(P \uparrow Q)$ if there is a particular reason to do so
    - The most common particular reasons would be
      - An explicit statement of the speaker to the effect that "not both"
      - Background knowledge that justifies this
        - » as in the election example above
      - Charity
        - » If the validity of an argument requires that the speaker mean "not both," one might note that rather than just say that the argument is invalid.

Necessary & Sufficient Conditions

- Conditional statements can express two kinds of condition:
  - Necessary conditions
    - E.g., "It is a necessary condition of being President of the United States that one be at least 35 years of age."
    - For "N is a necessary condition of Q", write (Q → N).
  - Sufficient conditions
    - E.g., "It is a sufficient condition of being elected President of the United States that one receive a majority of votes in the Electoral College."
    - Sometimes conditionals state only relative necessity: "If this paper is combustible, then it will catch fire when I put this match to it" (provided there is oxygen in the room, &c.).
    - For "S is the sufficient condition of Q", write (S → Q).
    - For necessary & sufficient conditions, write (C ↔ Q).

The Truth of Compound Statements

- Two kinds of statement are truth-functional:
  - i.e., the truth of the compound is a function of (= is determined by) the truth of the components
  - (P ∧ Q) is
    - *true* when both P and Q are true &
    - *false* otherwise.
  - (P ∨ Q) is
    - *true* when either P and Q are true &
    - *false* only if both are false.
- What about (P → Q)? Is it also a truth-functional connective?

These can be defined by truth-tables

P	Q	(P∧Q)
1	1	1
1	0	0
0	1	0
0	0	0

P	Q	(P∨Q)
1	1	1
1	0	1
0	1	1
0	0	0

P	Q	(P→Q)
1	1	?
1	0	?
0	1	?
0	0	?

if, →, & Truth-functionality

- When *if* is interpreted as a truth-functional connective, it is said to be
  - *false* if the antecedent is true & the consequent is false
  - *true* otherwise
    - i.e., true whenever the consequent is true
    - & true whenever the antecedent is false
      - *ex falso quodlibet*
- But that seems wrong
  - Consider these two statements:
    1. If I'm invisible, then everyone can see me.
    2. If I'm invisible, then no one can see me.
  - On the T-functional interpretation, they are both true.
  - But obviously (1) is false.
  - On the alternative, connectional, interpretation, the conditional is true if the consequent can be inferred from the antecedent (& acceptable, but unspecified, background assumptions).

P	Q	(P→Q)
1	1	1
1	0	0
0	1	1
0	0	1

if, →, & Truth-functionality (cont'd.)

- Still, the truth-functional version is useful, so we will write (P→ Q) when we read *if* & call it *material implication*.
  - This will not create a problem, since whenever the connectional "if P, then Q" is true, (P→ Q) is true also.
  - One must be more careful about the opposite: Proving that (P→ Q) and then asserting that one has shown "if P, then Q". The only way to show that is to give an argument from P to Q.