3 Linear Programming: A Geometric Approach

• Graphing Systems of Linear Inequalities in Two Variables
• Linear Programming Problems
• Graphical Solution of Linear Programming Problems
• Sensitivity Analysis (Optional)
Graphing Linear Inequalities

Ex. Graph $y \leq 2x - 1$

Notice that the line (=) is part of the solution

Also, any point in the lower half-plane satisfies the inequality so this region is shaded.
Graphing Linear Inequalities

Ex. Graph \( y \leq 2x - 1 \)

The line (=) is not part of the solution so it is dashed.

Any point in the lower half-plane satisfies the inequality.
Procedure for Graphing Linear Inequalities

1. Draw the graph of the equation obtained by replacing the inequality symbol with an equal sign. Make the line dashed if the inequality symbol is $<$ or $>$, otherwise make it solid.

2. Pick a test point in one of the half planes, substitute the $x$ and $y$ values into the inequality (use the origin whenever possible).

3. If the inequality is satisfied, shade the half plane containing the test point otherwise shade the other half plane.
Ex. Graph $3x + 2y < 6$

$3x + 2y = 6$

Dashed since $<$

Test $(0,0)$: $3(0) + 2(0) < 6$

True so shade region containing $(0,0)$
Ex. Graph $x \geq 2$

Ex. Graph $y < -3$
Ex. Graph the solution set for the system:

\[ \begin{align*}
3x + 2y &\leq 6 \\
3x + 2y &\geq 6
\end{align*} \]

Points must satisfy *both* inequalities (overlap of individual shaded regions)
Bounded Region

Unbounded Region
Linear Programming Problem

A *linear programming problem* consists of a linear objective function to be maximized or minimized, subject to certain constraints in the form of linear equalities or inequalities.
Ex. A small company consisting of two carpenters and a finisher produces and sells two types of tables: type \( A \) and type \( B \). Each type-\( A \) table will result in a profit of $50, and each type-\( B \) table will result in a profit of $54. A type-\( A \) table requires 3 hours of carpentry and 1 hour of finishing. A type-\( B \) table requires 2 hours of carpentry and 2 hours of finishing. Each day there are 16 hours available for carpentry and 8 hours available for finishing. How many tables of each type should be made each day to maximize profit? (Formulate but do not solve the linear programming problem.)
Organize the Information:

<table>
<thead>
<tr>
<th></th>
<th>Table $A$</th>
<th>Table $B$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>3</td>
<td>2</td>
<td>16 hours</td>
</tr>
<tr>
<td>Finishing</td>
<td>1</td>
<td>2</td>
<td>8 hours</td>
</tr>
<tr>
<td>Profit/table</td>
<td>$50</td>
<td>$54</td>
<td></td>
</tr>
</tbody>
</table>

Let $x =$ # type $A$ and $y =$ # type $B$.

The objective function to be maximized is given by:

$$P = 50x + 54y$$
The constraints are given by:

\[
\begin{align*}
\text{Carpentry} & \quad 3x + 2y \leq 16 \\
\text{Finishing} & \quad x + y \leq 8
\end{align*}
\]

Also the number of units is nonnegative:

\[
\begin{align*}
x & \geq 0, \quad y \geq 0
\end{align*}
\]

So we have:

Maximize \( P = 50x + 54y \)

Subject to: \( 3x + 2y \leq 16 \)

\[
\begin{align*}
x + y & \leq 8 \\
x & \geq 0, \quad y \geq 0
\end{align*}
\]
Ex. A particular company manufactures specialty chairs in two plants. Plant I has an output of at most 150 chairs/month. Plant II has an output of at most 120 chairs/month. The chairs are shipped to 3 possible warehouses - A, B, and C. The minimum monthly requirements for warehouses A, B, and C are 70, 70, and 80, respectively. Shipping charges from plant I (to A, B, and C) are $30, $32, and $38/chair and from plant II (to A, B, and C) are $32, $28, $26. How many chairs should be shipped to each warehouse to minimize the monthly shipping cost? (Formulate but do not solve the linear programming problem.)
Organize the Information:

Number of Chairs

<table>
<thead>
<tr>
<th>Plant</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>Max. Prod.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>150</td>
</tr>
<tr>
<td>II</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
<td>120</td>
</tr>
<tr>
<td>Min.</td>
<td>70</td>
<td>70</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

Cost to Ship

<table>
<thead>
<tr>
<th>Plant</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>30</td>
<td>32</td>
<td>38</td>
</tr>
<tr>
<td>II</td>
<td>32</td>
<td>28</td>
<td>26</td>
</tr>
</tbody>
</table>

...
We want to minimize the cost function:

$$C = 30x_1 + 32x_2 + 38x_3 + 32x_4 + 28x_5 + 26x_6$$

Production constraints:

Plant I \quad x_1 + x_2 + x_3 \leq 150

Plant II \quad x_4 + x_5 + x_6 \leq 120

Warehouse constraints:

A \quad x_1 + x_4 \geq 70

B \quad x_2 + x_5 \geq 70

C \quad x_3 + x_6 \geq 80
So the problem is:

Minimize: \( C = 30x_1 + 32x_2 + 38x_3 + 32x_4 + 28x_5 + 26x_6 \)

Subject to: \( x_1 + x_2 + x_3 \leq 150 \)
\( x_4 + x_5 + x_6 \leq 120 \)
\( x_1 + x_4 \geq 70 \)
\( x_2 + x_5 \geq 70 \)
\( x_3 + x_6 \geq 80 \)
\( x_1 \geq 0, x_2 \geq 0, \ldots, x_6 \geq 0 \)
Theorem: Linear Programming

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set associated with the problem.

If the objective function is optimized at two adjacent vertices of the feasible set, then it is optimized at every point on the line segment joining these vertices (hence infinitely many solutions).
The Method of Corners

1. Graph the feasible set.

2. Find the coordinates of all corner points (vertices) of the feasible set.

3. Evaluate the objective function at each corner point.

4. Find the maximum (minimum). If there is only one, it is unique. If two adjacent vertices share this value, there are infinitely many solutions given by the points on the line segment connecting these vertices.
Graphical Method (2 variables)

Maximize \( P = 4x + 5y \)
Subject to \( 3x + 5y \leq 20 \)
\( x + y \leq 6 \)
\( x \geq 0, y \geq 0 \)

Optimal solution (if it exists) will occur at a corner point

Check Vertices:

- \((0, 0) : P = 4(0) + 5(0) = 0\)
- \((0, 4) : P = 4(0) + 5(4) = 20\)
- \((5, 1) : P = 4(5) + 5(1) = 25\)
- \((6, 0) : P = 4(6) + 5(0) = 24\)

The maximum is 25 at \((5, 1)\)
Ex.  Minimize  \[ C = 10x + 11y \]
Subject to  \[ 20x + 10y \geq 300 \]
\[ 15x + 15y \geq 300 \]
\[ 10x + 20y \geq 250 \]
\[ x \geq 0, \ y \geq 0 \]

Check Vertices:
\[ (0,30) : C = 330 \]
\[ (10,10) : C = 210 \]
\[ (15,5) : C = 205 \]
\[ (25,0) : C = 250 \]

The minimum is 205 at (15, 5).
Ex. Unbounded – No Solution
Maximize $P = 3x + 5y$
Subject to $3x - y \geq -1$
$x - 2y \leq 2$
$x \geq 0, y \geq 0$

Unbounded Feasible set
No Maximum
Ex. Unfeasible Problem

Maximize \( P = 10x + 11y \)

Subject to

\( 3x + 4y \leq 1 \)
\( 2x + y \geq 4 \)
\( x \geq 0, y \geq 0 \)

No Feasible set – No overlap
Example
A toy manufacturer makes bikes, for a profit of $4, and wagons, for a profit of $5. To produce a bike requires 3 hours of machine time and 4 hours of painting time. To produce a wagon requires 5 hours machine time and 4 hours of painting time. There are 20 hours of machine time and 24 hours of painting time available per day. How many of each toy should be produced to maximize the profit?
Solution

Let \( x \) = the number of bikes produced
Let \( y \) = the number of wagons produced
The linear programming problem is

Maximize  \( P = 4x + 5y \)  
Subject to:  
\[ 3x + 5y \leq 20 \]
\[ 4x + 4y \leq 24 \]
\[ x \geq 0, \ y \geq 0 \]

Maximize  \( P = 4x + 5y \)  
Subject to:  
\[ 3x + 5y \leq 20 \]
\[ x + \ y \leq 6 \]
\[ x \geq 0, \ y \geq 0 \]

\[ \ldots \]
Solution (cont.)

Sketch the graph of the feasible set to locate all of the corner points.

In order to produce the maximum profit $25, the toy manufacturer needs to make 5 bikes and 1 wagon.

Check Vertices:

(0,0) : $P = 4(0) + 5(0) = 0$
(0,4) : $P = 4(0) + 5(4) = 20$
(5,1) : $P = 4(5) + 5(1) = 25$
(6,0) : $P = 4(6) + 5(0) = 24$
Sensitivity Analysis

Investigating how changes in the parameters of a linear programming problem affect its optimal solution.

Check Changes in:

1. Coefficients of the Objective Function.
Ex. Changes in Coefficients of the Objective Function

Maximize \( P = 4x + 5y \)
Subject to \( 3x + 5y \leq 20 \)
\( x + y \leq 6 \)
\( x \geq 0, \ y \geq 0 \)

Examine the first variable \( P = cx + 5y \)

which is \( y = \frac{-c}{5}x + \frac{P}{5} \)

Compare this slope to the slopes of the constraints

max is 25 at (5, 1)
The optimal solution will remain unaffected as long as

\[-\frac{c}{5} \leq -\frac{3}{5} \quad \text{and} \quad -1 \leq -\frac{c}{5}\]

\[c \geq 3 \quad \text{and} \quad 5 \geq c\]

\[3 \leq c \leq 5\]

So if the value of the coefficient of $x$ stays between 3 and 5, then the optimal solution is still at (5, 1).
Ex. Changes in Constants on the Right-Hand Side of the Constraints

Maximize \( P = 4x + 5y \)
Subject to \( 3x + 5y \leq 20 \)
\( x + y \leq 6 \)
\( x \geq 0, y \geq 0 \)

Examine the first constraint \( 3x + 5y \leq 20 \)

Add \( h \) \( 3x + 5y \leq 20 + h \)

This is parallel to the first constraint.

max is 25 at (5, 1)
Find the intersection point of

\[ 3x + 5y = 20 + h \quad \text{and} \quad x + y = 6 \]

\[ x = \frac{1}{2}(10 - h), \quad y = \frac{1}{2}(2 + h) \]

Using \( x \geq 0 \) and \( y \geq 0 \):

\[ x = \frac{1}{2}(10 - h) \geq 0 \Rightarrow h \leq 10 \]

\[ y = \frac{1}{2}(2 + h) \geq 0 \Rightarrow h \geq -2 \]

So for a meaningful solution, the first constraint must be between \( 20 - 2 = 18 \) and \( 20 + 10 = 30 \).
For example, since

\[ x = \frac{1}{2}(10 - h) \text{ and } y = \frac{1}{2}(2 + h) \]

If \( h = 2 \), then the constraint becomes \( 3x + 5y \leq 22 \)

\[ x = \frac{1}{2}(10 - 2) = 4 \]

\[ y = \frac{1}{2}(2 + 2) = 2 \]

\[ P = 4(4) + 5(2) = 26 \]

Therefore increasing the constraint from 20 to 22 results in an increase of \( P \) from 25 to 26.
Shadow Prices

The *shadow price* of the $i^{th}$ resource is the amount by which the objective function is improved if the right-hand side of the $i^{th}$ constraint is increased by 1 unit.
From the last example, since

\[ x = \frac{1}{2}(10 - h) \quad \text{and} \quad y = \frac{1}{2}(2 + h) \]

Plugging into \( P = 4x + 5y \) we get

\[
= 4 \left( \frac{1}{2} (10 - h) \right) + 5 \left( \frac{1}{2} (2 + h) \right)
= 25 + 0.5h
\]

Since the maximum for the original problem was 25, this shows that for each unit increase of the constant in the first constraint there is a 0.5 increase in \( P \).